

TWISTS AND SPECTRAL TRIPLES FOR ISOSPECTRAL DEFORMATIONS.

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ABSTRACT. We construct explicitly the symmetries of the isospectral deformations as twists of Lie algebras and demonstrate that they are isometries of the deformed spectral triples.

1. INTRODUCTION

The isospectral deformations, which have been introduced by Connes and Landi [4], with the examples of the noncommutative 3 and 4 spheres, have attracted recently much attention.

The deformation itself, which, looking only at the algebra is of the "star deformation" (Moyal) type, has been discussed in other contexts and the construction of similar, physically motivated examples can be found in recent works by Kulish and Mudrov [9] as well as in Paschke Ph.D. thesis [12].

On the mathematical side (though also with a physical motivation) it could be traced back to sixties [6] or, more recently, to the works of Rieffel [14, 15, 16], Dubois-Violette [7] and Mourre [8].

However, only after [4] it became apparent that these kind of deformation allows to describe a real noncommutative spin manifold in the sense of real spectral triples. Moreover, their geometry is distinct from the most of other known deformations allowing for definition of C^∞ elements, no dimension drop and the transfer of most geometric features from the commutative case [5].

In this paper we discuss the notion of symmetries of the spectral triples for isospectral deformations. The existence of quantum group symmetries for isospectral deformations has been first noted by Connes and Dubois-Violette [5]. Here, for the purpose of making connections with the work on symmetries of spectral triples [13] we have independently developed a dual approach (using Lie algebras and not Lie groups). We demonstrate that the twist by a Cartan subalgebra provides a deformed symmetry of the *isospectral deformation* and that it is an isometry of the deformed manifold.

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2. SYMMETRIES OF THE ISOSPECTRAL DEFORMATIONS

The isospectral deformation has appeared in the construction of the examples of noncommutative 4 spheres, which have the same instanton bundle as the "classical" sphere. One of the crucial questions posed by the construction was whether the constructed *spheres* still are symmetric, i.e. whether the natural $SO(5)$ symmetries (or, respectively, $SO(4)$ for the 3-sphere) are preserved (in the form of deformed Hopf algebras) and whether the constructed spectral triples are *symmetric* in the sense of [13].

Here, we shall present the symmetry in terms of the deformation of the universal enveloping algebra acting on the deformed algebra. We shall use both a general approach as well as the description in terms of generators and relations, the latter to make connections with the known specific examples. This enabled us to make connection [17] with the work of Rieffel [14, 15] and to present the isospectral deformation (on the algebraic level) as a special case of the strict deformation quantization as described elsewhere [16].

Let H be a Hopf algebra (symmetry algebra), which contains two independent, mutually commuting generators of $U(1)$ symmetries, i.e. operators h_1, h_2 such that:

$$[h_1, h_2] = 0, \quad \Delta h_i = 1 \otimes h_i + h_i \otimes 1.$$

Consider an algebra \mathcal{A} on which H acts from the left. The generators h_i act on \mathcal{A} as generators of mutually independent $U(1)$ symmetries.

Remark 1. With respect to the action of h_1, h_2 we can select out elements in \mathcal{A} of degree (n_1, n_2) , we say that $t \in \mathcal{A}$ is of degree (n_1, n_2) iff:

$$(1) \quad h_1 \triangleright t = n_1 t, \quad h_2 \triangleright t = n_2 t.$$

Remark 2. The product in the algebra \mathcal{A} can be deformed, first on elements of given degree, and then extending the deformation by linearity:

$$(2) \quad a * b = ab\lambda^{n_1^a n_2^b},$$

where λ is a complex number such that $|\lambda| = 1$.

This is the crucial step of the construction of the isospectral deformation as described in [4], for future reference we shall denote the deformed algebra by \mathcal{A}_λ .

This deformation, can be extended, in the case of a differential manifold and a Lie algebra acting on the differential functions, to all C^∞ functions [4].

It would be useful to introduce a *quantization map*:

$$\mathcal{A} \ni a \mapsto \underline{a} \in \mathcal{A}_\lambda,$$

so that the Eq.2 could be rewritten as:

$$(3) \quad \underline{a} * \underline{b} = \underline{ab} \lambda^{n_1^a n_2^b},$$

Before we proceed with the deformation of the symmetry and the generalization of the above deformation, let us observe that the star structure of the algebra \mathcal{A}_λ could be also deformed:

$$(4) \quad (\underline{a})^* = \lambda^{n_1^a n_2^a} (\underline{a^*}),$$

2.1. Example: Lie algebras of rank $n \geq 2$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of a Lie algebra \mathfrak{g} with the Cartan matrix a_{ij} , and the set of generators in the Chevalley basis: h_i , which commute with each other (we assume that the rank of the Lie algebra is at least 2).

$$(5) \quad [h_i, h_j] = 0,$$

and x_k^\pm , which satisfy:

$$(6) \quad [h_i, x_j^\pm] = \pm a_{ij} x_j^\pm.$$

$$(7) \quad [x_i^+, x_j^-] = \delta_{ij} h_i,$$

together with the Serre relations:

$$(8) \quad \sum_{k=0}^{1-a_{ij}} \binom{1-a_{ij}}{k} (x_i^+)^k x_j^+ (x_i^+)^{1-a_{ij}-k} = 0.$$

Then, choosing, for instance, h_1, h_2 as the generators discussed earlier we have the desired situation.

3. THE TWISTED SYMMETRY H_λ

We shall construct here the deformation of the symmetry algebra H_λ , which is the twist (cocycle deformation) of H . Furthermore, we shall verify in the next section that this will be the symmetry of the deformed algebra \mathcal{A}_λ . It will have the same algebra structure and the same action of the generators on the elements of \mathcal{A}_λ , with the coalgebra and antipode modified by a twist. For a general theory, examples and details of twists see [1] and [10]. We shall use capital letters to denote the elements of the deformed algebra, then, if no misinterpretation is possible we shall write Δ instead of Δ_λ .

Definition 1. Let us define H_Ψ as an algebra isomorphic to H , however, with a twisted coproduct:

$$(9) \quad \Delta_\lambda T = \Psi \Delta t \Psi^{-1},$$

where Ψ is an invertible element of $H \otimes H$, which satisfies a cocycle condition:

$$(10) \quad \Psi_{12}(\Delta \otimes \text{id})\Psi = \Psi_{23}(\text{id} \otimes \Delta)\Psi,$$

$$(11) \quad (\epsilon \otimes \text{id})\Psi = 1 = (\text{id} \otimes \epsilon)\Psi.$$

In the particular example associated with the isospectral deformations we shall take an element Ψ_c associated with the Cartan subalgebra generated by h_1, h_2 :

$$H \otimes H \ni \Psi_c = \lambda^{-H_1 \otimes H_2},$$

and we shall call this particular twist H_λ . The deformed coproduct in H_λ becomes:

$$(12) \quad \Delta_\lambda T = \lambda^{-H_1 \otimes H_2}(\Delta t)\lambda^{H_1 \otimes H_2},$$

To see that Ψ_c satisfies the cocycle condition we shall verify it explicitly:

$$\begin{aligned} \Psi_{12}(\Delta \otimes \text{id})\Psi &= \lambda^{-H_1 \otimes H_2 \otimes 1}(\Delta \otimes \text{id})\lambda^{-H_1 \otimes H_2} = \\ &= \lambda^{-H_1 \otimes H_2 \otimes 1}\lambda^{-(1 \otimes H_1 + H_1 \otimes 1) \otimes H_2} = \\ &= \lambda^{-H_1 \otimes H_2 \otimes 1 - 1 \otimes H_1 \otimes H_2 - H_1 \otimes 1 \otimes H_2}. \end{aligned}$$

on the other hand:

$$\begin{aligned} \Psi_{23}(\text{id} \otimes \Delta)\Psi &= \lambda^{-1 \otimes H_1 \otimes H_2}(\text{id} \otimes \Delta)\lambda^{-H_1 \otimes H_2} = \\ &= \lambda^{-1 \otimes H_1 \otimes H_2}\lambda^{-H_1 \otimes (1 \otimes H_2 + H_2 \otimes 1)} = \\ &= \lambda^{-H_1 \otimes H_2 \otimes 1 - 1 \otimes H_1 \otimes H_2 - H_1 \otimes 1 \otimes H_2}. \end{aligned}$$

which proves (10). ■

The counit does not change:

$$(13) \quad \epsilon(T) = \epsilon(t),$$

whereas the antipode is twisted by an element U :

$$(14) \quad U = \Psi_{(1)}S(\Psi_{(2)}),$$

$$(15) \quad S(T) = US(t)U^{-1}.$$

3.1. **Twist of $U_\lambda(\mathfrak{g})$.** In the particular example of the Lie algebra \mathfrak{g} , we have an algebra with the same relations as (5-8). For simplicity we shall call $a_{1i} = \alpha_i$ and $a_{2i} = \beta_i$.

Then, using (12) we might calculate explicitly:

$$(16) \quad \begin{aligned} \Delta H_i &= H_i \otimes 1 + 1 \otimes H_i, \\ \Delta X_i^\pm &= \lambda^{-H_1 \otimes H_2} (X_i^\pm \otimes 1 + 1 \otimes X_i^\pm) \lambda^{-H_1 \otimes H_2} = \\ &= X_i^\pm \otimes \lambda^{\mp \alpha_i H_2} + \lambda^{\mp \beta_i H_1} \otimes X_i^\pm, \end{aligned}$$

The obtained object is a triangular Hopf algebra with the universal \mathcal{R} -matrix:

$$\mathcal{R} = \lambda^{H_2 \otimes H_1 - H_1 \otimes H_2}.$$

For completeness we give here the counit and the antipode calculated using (15):

$$(17) \quad \begin{aligned} \epsilon(H_i) &= 0, & \epsilon(X_i^\pm) &= 0, \\ SH_i &= -H_i, & SX_i^\pm &= -\lambda^{\pm \beta_i H_1} X_i^\pm \lambda^{\pm \alpha_i H_2}, \end{aligned}$$

4. TWISTED SYMMETRY OF THE ALGEBRA \mathcal{A}_λ

We shall demonstrate now that the twisted Hopf algebra defined in the previous section is the symmetry algebra of the deformation \mathcal{A}_λ .

Even more, to see it, we shall use the generalization of the deformation of \mathcal{A} by an arbitrary twist Ψ .

Definition 2. Let us define a deformed product for elements of \mathcal{A} (we shall denote the deformed algebra \mathcal{A}_Ψ): If $\underline{a}, \underline{b} \in \mathcal{A}_\Psi$ then:

$$(18) \quad \underline{m}(\underline{a} \otimes \underline{b}) = \underline{m}(\Psi^{-1} \triangleright (a \otimes b)),$$

where m is the multiplication map

$$m : \mathcal{A} \otimes \mathcal{A} \ni a \otimes b \rightarrow ab \in \mathcal{A}.$$

and \underline{m} is the multiplication in \mathcal{A}_Ψ :

$$\underline{m} : \mathcal{A}_\Psi \otimes \mathcal{A}_\Psi \ni \underline{a} \otimes \underline{b} \rightarrow \underline{a} * \underline{b} \in \mathcal{A}_\Psi.$$

We shall verify explicitly this defines an associative product:

$$\begin{aligned} \underline{a} * (\underline{b} * \underline{c}) &= \underline{m}(\Psi^{-1} \triangleright (a \otimes (\underline{b} * \underline{c}))) = \\ &= \underline{m}((\text{id} \otimes \Delta) \Psi^{-1}) \Psi_{23}^{-1} \triangleright (a \otimes b \otimes c) = \dots \end{aligned}$$

and using the cocycle condition for Ψ ;

$$\begin{aligned} \dots &= \dots ((\Delta \otimes \text{id}) \Psi^{-1}) \Psi_{12}^{-1} \triangleright (a \otimes b \otimes c) = \\ &= (\underline{a} * \underline{b}) * \underline{c}, \end{aligned}$$

■

Definition 3. The action of the twisted symmetry H_Ψ on \mathcal{A}_Ψ could be defined as the twisting of the action of H on \mathcal{A} . If $\underline{a} \in \mathcal{A}_\Psi$, and $T \in H_\Psi$ then:

$$(19) \quad T \triangleright \underline{a} = \underline{(t \triangleright a)},$$

where T is deformed t .

To verify that this is an action of the Hopf algebra on \mathcal{A}_Ψ we have to verify first the compatibility with the product.

We shall present here a general proof, which we shall repeat later in the specific case of $U_\lambda(\mathfrak{g})$, using generators and homogeneous elements. The general proof uses the twisting relation (12).

Proof: We have, for arbitrary $T \in H_\Psi$ (which we identify with $t \in H$):

$$(20) \quad \begin{aligned} T \triangleright (\underline{a} * \underline{b}) &= \\ &= t \triangleright m(\Psi^{-1} \triangleright (a \otimes b)) = \\ &= \underline{m(((t_{(1)} \otimes t_{(2)}) \Psi^{-1}) \triangleright (a \otimes b))}, \end{aligned}$$

where we have used the form of the product (18) the definition of the action (19) and the undeformed coproduct of t . On the other hand we have:

$$(21) \quad \begin{aligned} T \triangleright (\underline{a} * \underline{b}) &= \\ &= (T_{(1)} \triangleright \underline{a}) * (T_{(2)} \triangleright \underline{b}) = \\ &= \underline{m((\Psi(t_{(1)} \otimes t_{(2)}) \Psi^{-1}) \triangleright (\underline{a} \otimes \underline{b}))} = \\ &= \underline{m((\Psi^{-1} \Psi(t_{(1)} \otimes t_{(2)}) \Psi^{-1}) \triangleright (a \otimes b))} = \\ &= \underline{m(((t_{(1)} \otimes t_{(2)}) \Psi^{-1}) \triangleright (a \otimes b))}, \end{aligned}$$

which ends the proof. ■

Remark 3. In particular, H_λ is the symmetry of the deformed \mathcal{A}_λ algebra.

5. THE STAR STRUCTURE

Suppose that H is a star Hopf algebra and that the action on \mathcal{A} is compatible with the star structures on both algebras:

$$(22) \quad t \triangleright a^* = ((St)^* \triangleright a)^*.$$

Lemma 1. The twisted algebra H_Ψ is a star Hopf algebra provided that $\Psi^* = \Psi^{-1}$ (which for the twist by the Cartan subalgebra translates to: $H_1^* \otimes H_2^* = H_1 \otimes H_2$).

This follows directly from (12).

We shall prove that the action is compatible with the star structure (4), however to prove it we shall rewrite (4) in a more general way:

$$(23) \quad \underline{a}^* = \underline{(U^{-1} \triangleright a)^*},$$

where U is as defined in (14).

$$(24) \quad \begin{aligned} T \triangleright \underline{a}^* &= T \triangleright ((U^{-1} \triangleright a)^*) = \\ &= t \triangleright \underline{(U^{-1} \triangleright a)^*} = \\ &= \underline{(St)^* U^{-1} \triangleright a)^*} \end{aligned}$$

on the other hand:

$$(25) \quad \begin{aligned} T \triangleright \underline{a}^* &= ((ST)^* \triangleright \underline{a})^* = \\ &= ((U(St)U^{-1})^* \triangleright \underline{a})^* = \\ &= \left(\underline{U(St)^* U^{-1} \triangleright a} \right)^* = \\ &= \underline{(U^{-1} U(St)^* U^{-1} \triangleright a)^*} = \\ &= \underline{(St)^* U^{-1} \triangleright a)^*}. \end{aligned}$$

where we have used $U^{-1} = U^*$. This follows from $\Psi^* = \Psi^{-1}$.

5.1. The action of $U_\lambda(\mathfrak{g})$. For completeness we mention that one could always give the definition (19) using the generators alone:

$$(26) \quad X_i^\pm \triangleright \underline{T} = \underline{x_i^\pm \triangleright T},$$

$$(27) \quad H_i \triangleright \underline{T} = \underline{h_i \triangleright T},$$

and then extend it to the whole of $U_\lambda(\mathfrak{g})$ for every $T \in \mathcal{A}_\lambda$.

To verify the compatibility of this action with the deformed product in \mathcal{A}_λ clearly only action of X_i^\pm must be checked, as the coproduct of H_i remains not changed. Before we start let us observe:

Remark 4. If $T \in \mathcal{A}$ is homogeneous with degree (n_1, n_2) then $x_i^\pm \triangleright T$ is also homogeneous with degree $(n_1 \pm a_{1i}, n_2 \pm a_{2i})$. This follows directly from (6).

Proof: Consider $X_i^\pm \triangleright (\underline{a} * \underline{b})$, where a, b are homogeneous elements of \mathcal{A} . On one hand:

$$(28) \quad X_i^\pm \triangleright (\underline{a} * \underline{b}) = \lambda^{n_1^a n_2^b} \left(\underline{x_i^\pm \triangleright (ab)} \right),$$

on the other hand, calculating it directly:

$$\begin{aligned}
(29) \quad X_i^\pm \triangleright (\underline{a} * \underline{b}) &= (X_i^\pm \triangleright \underline{a}) * (\lambda^{\mp \alpha_i H_2} \triangleright \underline{b}) + (\lambda^{\mp \beta_i H_1} \triangleright \underline{a}) * (X_i^\pm \triangleright \underline{b}) \\
&= \overline{(x_i^\pm \triangleright a) * b \lambda^{\mp \alpha_i n_2^b} + \lambda^{\mp \beta_i n_1^a} \underline{a} * (x_i^\pm \triangleright b)} \\
&= \overline{\lambda^{\mp \alpha_i n_2^b} \lambda^{(n_1^a \pm \alpha_i) n_2^b} (x_i^\pm \triangleright a) b + \lambda^{\mp \beta_i n_1^a} \lambda^{n_1^a (n_2^b \pm \beta_i)} a (x_i^\pm \triangleright b)} \\
&= \lambda^{n_1^a n_2^b} \overline{(x_i^\pm \triangleright (ab))}.
\end{aligned}$$

■

6. DIFFERENTIAL STRUCTURES

Differential structures on twisted Hopf algebras (and their duals) has been extensively studied by Majid and Oeckl [11], where the stability of bicovariant calculi under twisting has been shown. We briefly restate these results for the particular situation of the differential calculi over \mathcal{A}_Ψ invariant under the action of H_Ψ .

Let $\Omega(\mathcal{A})$ be the differential algebra of forms over the algebra \mathcal{A} . We shall assume that the $\Omega(\mathcal{A})$ is invariant with respect to the action of H , i.e. the action of H extends on $\Omega(\mathcal{A})$ and intertwines the exterior derivative:

$$(30) \quad H \ni t \triangleright (d\omega) = d(t \triangleright \omega),$$

By using the same procedure as in the case of the deformation of \mathcal{A} we simply deform $\Omega(\mathcal{A})$ by modifying the product using (18). In particular, for two forms of homogeneous degree (defined similarly as for the elements of \mathcal{A}) we have:

$$(31) \quad \underline{\omega} \wedge \underline{\rho} = \lambda^{n_\omega^\perp n_\rho^2} \underline{(\omega \wedge \rho)},$$

thus extending the quantization map to the differential algebra.

First we shall prove that on $\Omega(\mathcal{A}_\Psi)$ there exists an exterior derivative making it a differential algebra.

Lemma 2. The exterior derivative defined as:

$$(32) \quad d\underline{\omega} = \underline{d\omega},$$

makes $\Omega(\mathcal{A}_\Psi)$ a differential algebra.

We need to verify the Leibniz rule (it is obvious that $d^2 \equiv 0$). This follows, however, directly from the graded Leibniz rule of the undeformed differential algebra and the relation (32).

Lemma 3. The deformed symmetry algebra acts on the quantized differential complex $\Omega(\mathcal{A}_\Psi)$.

It remains only to verify that the action commutes with the external derivative.

$$\begin{aligned} T \triangleright \underline{d\omega} &= \underline{t \triangleright d\omega} = \\ &= \underline{d(t \triangleright \omega)}. \end{aligned}$$

and using (32) we see that the action of T is well-defined.

Similarly one verifies that the star structure (if defined for the original differential complex) is preserved.

7. SPECTRAL TRIPLES

Let us assume that there exists a spectral triple for the algebra \mathcal{A} , i.e. we have all the data : Hilbert space \mathcal{H} on which \mathcal{A} is represented as bounded operators, the Dirac operator D (grading γ in case of even dimensions and J for real spectral triples) satisfying all the axioms as defined in [3]. Additionally, we assume that the spectral triple is *symmetric* (as defined in [13]), so that there exists the action of an Hopf algebra H on \mathcal{A} , and that the crossproduct of H and \mathcal{A} is represented on the Hilbert space. We call H isometry if the representation of H commutes with D .

Now we can state the main lemma:

Lemma 4. The deformation of the algebra \mathcal{A} by a twist Ψ by a Cartan subalgebra allows for the representation of \mathcal{A}_Ψ and the crossproduct of H_Ψ with \mathcal{A}_Ψ on the same Hilbert space. Moreover, with the Dirac operator D , as taken from the undeformed spectral triple we shall have a spectral triple for \mathcal{A}_Ψ , which, moreover, will be invariant under the twisted Hopf algebra H_Ψ .

The sketch of the construction of the deformed spectral triple has been given in [4] (Theorem 6), here we only need to verify that H_Ψ is the symmetry of algebra.

We need to define the representation of \mathcal{A}_Ψ on the Hilbert space, which gives rise to the representation of the crossproduct.

For $v \in \mathcal{H}$, $\underline{a} \in \mathcal{A}_\Psi$ we have:

$$(33) \quad \underline{a}v = \mu(\Psi^{-1}(a \otimes v)),$$

where μ denotes the representation of \mathcal{A} , $\mu(a \otimes v) = av$ Clearly, for $L \in H_\Psi$, $a \in \mathcal{A}_\Psi$ and $v \in \mathcal{H}$ we have:

$$(34) \quad L(\underline{a}v) = l\mu(\Psi^{-1}(a \otimes v)) = \mu((\triangle l)\Psi^{-1}(a \otimes v)),$$

where we have used the known form of the representation of the H_Ψ , which is the same as this of H (l is "undeformed" L). On the other hand:

$$L(\underline{a}v) = \underline{\mu} \left(\Psi(\triangle l) \Psi^{-1} \triangleright (\underline{a} \otimes v) \right),$$

where $\underline{\mu}$ is the representation of the deformed algebra $\mu(\underline{a} \otimes v) = \underline{a}v$. But using (33) we obtain the same result as in (34). ■

Since the representation of H_Ψ is then the same as Ψ and D does not change as well, it follows at once that H_Ψ will be the isometry of the deformed spectral triple.

8. CONCLUSIONS

As we have shown, the isospectral deformation has a Hopf algebra isometry, which is a twist of the classical Lie algebra symmetry of the commutative space. Some physical models based on twists have already been described in the literature, for instance, the twisted Lorenz group and Minkowski space, which has been discussed by Kulish and Mudrov in [9].

Since the algebraic form of the symmetries remains intact and only the coproduct changes one could expect modifications with respect to "undeformed physics" not on the one-particle level but only on the level of interactions. It remains to be verified, especially for gauge theories, whether such deformations are physically plausible.

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